

# PERMUTATION TECHNIQUES BASED ON EUCLIDEAN ANALYSIS SPACES: A NEW AND POWERFUL STATISTICAL METHOD FOR ECOLOGICAL RESEARCH

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**Abstract.** The statistical procedures that are most widely used in ecological population and community research belong to the family of parametric methods. Embedded in these procedures are assumptions about the normal distribution of the underlying population, homogeneity of variances and linear response patterns. One of the problems encountered in ecological and vegetation studies, however, is that these assumptions are very difficult, if not impossible, to meet. In addition, a very serious shortcoming of the most widely used statistical methods is the lack of congruence between the geometry of the data space, which is for the most part Euclidean, and the analysis space, which in the standard parametric tests and most of the nonparametric tests, is not Euclidean. In ecological and vegetation studies, the combination of a failure to meet model assumptions and a lack of congruence between the geometries of the data space and the analysis space can lead, as shown in this paper, to gross errors in data interpretation and hypothesis testing.

A new and powerful statistical technique (MRPP) is presented in this paper which is free from assumptions about the underlying distribution model of the population under analysis, can easily handle nonlinear data structures and more importantly meets the congruence principle (a common geometry for both the data and analysis spaces). The theoretical formulations for MRPP and its randomized block design counterpart MRBP as well as their relationship to other statistical methods are outlined in the first part of a paper. This is followed by computer algorithms and programs needed for their implementation as well as a series of detailed examples which demonstrate major advantages of MRPP and MRBP over the currently most widely used statistical methods.

## 1.0 Introduction

The strict application of the scientific method to ecology in general and vegetation science, in particular, is critical if both disciplines are to advance significantly our knowledge of natural and introduced ecosystems as well as remove the stigma of "soft science" that has been often applied to ecology (Simberloff 1981, Peters 1980). The full application of the scientific method not only implies the inductive route, that is what we generally call data interpretation, but more critically the hypothesis-deductive route. From the inductive approach we are able to generate hypotheses whose function are to explain in a cohesive and comprehensive way why a particular pattern in our observations has occurred. The next step involves the development of critical predictions from our general hypothesis that are to be experimentally tested (Fig. 1). In almost all the cases involving vegetation studies it is impractical and/or impossible to analyze the entire population. This situation has lead us to make inferences about ecosystem properties based upon a limited number of samples forcing us to rely heavily on statistical procedures. The statistical procedure therefore becomes a key component

in the acceptance or rejection of a hypothesis.

The statistical procedures that are widely used in ecology belong to the family of parametric methods. These methods assume an underlying distribution model for the data which is characterized by a given set of parameters. The test then involves the calculation of the parameters from the sampled observations and an estimation of the probability of obtaining that particular set of parameter values if all the samples had been taken from the same population (null hypothesis). The majority of the tests that are commonly available (e.g., t-test, Hotelling  $T^2$ , analysis of variance (ANOVA), multivariate analysis of variance (MANOVA), multiple regression, canonical correlation, discriminant analysis) involve the estimation and testing of means and are characterized by the following assumptions:

a. The underlying distribution model for a population is assumed to be normal or multivariate normal with equal (homogeneous) variance or an equal variance-covariance matrix. The normal distribution (and its counterpart the multivariate normal) are totally characterized by the mean and variance (mean vector and variance-covariance matrix). As a result of our as-

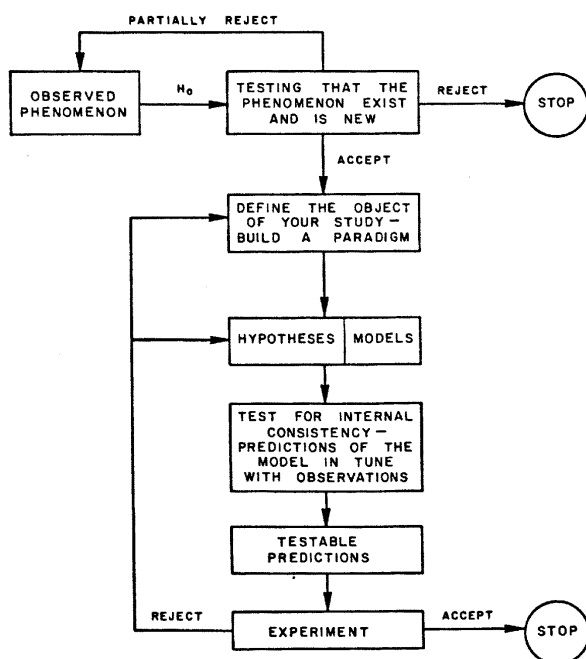


Fig. 1. A diagram showing the sequence of inductive and hypothetical-deductive steps on the scientific method. Note:  $H_0$  = null hypothesis.

sumption of equal variance (variance-covariance matrix), all the statistical procedures listed above (ANOVA, MANOVA, etc.) are mean based tests.

b. A second and very important assumption made in all the above mentioned tests is that the underlying distribution models are linear.

c. The methods used to calculate the pertinent statistics involve least squares. As a consequence, the inherent nature of the analysis space is non-metric.

In ecological field studies, all of these assumptions are very difficult if not impossible to meet. The underlying distribution model of the population, most of the time, is not known and in many cases is definitely not normal (for example, precipitation distribution in semi-arid regions is better approximated by an exponential distribution (Sala *et al.* 1982)). In the case of univariate analyses, this may be less of a problem because most univariate distributions can be brought closer to a normal distribution by an adequate transformation. However, this approach is difficult when multivariate distributions are involved. A most serious constraint, however, is the assumed equality of variances (variance-covariance matrix) of the underlying models. There are many cases of vegetation analyses when comparisons are made between vegetation communities that have different patterns of plant distribution (Fig. 2) or between particular vegetation treatments which may result in significant alterations in plant distribution patterns. Under these circumstances the assumption of equal variances (variance-covariance matrices) does not

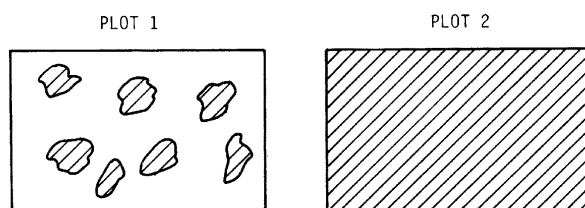


Fig. 2. Plot 1 and Plot 2 have equal biomass but different distribution, uniform in Plot 2 scatter in Plot 1.

hold.

The linear assumptions involved in ANOVA, MANOVA or other standard statistical tests may be very difficult to meet. As mentioned by Orlóci (1979), linear relationships are rarely met in ecological systems and furthermore most of the ecologically interesting data is intrinsically tied up with non-linear correlations.

The least square solution method that is used in the standard statistical methods (ANOVA, MANOVA, etc.) may present another serious problem in ecological research, this time because of major differences between the geometry of the data space and the geometry of the analysis space. Let us define  $\Delta_{ij}$  as the distance between points  $i$  and  $j$  in a given space. A space is called a metric space if (a)  $\Delta_{ji} \geq 0$  and  $\Delta_{ii} = 0$ ; (b)  $\Delta_{ij} = \Delta_{ji}$  (symmetry) and (c)  $\Delta_{ij} + \Delta_{jk} \geq \Delta_{ik}$  (triangle inequality). The most familiar metric space is the ordinary Euclidean space. All of the standard statistical methods belong to the family of generalized linear models where the estimation of parameters involve the minimiza-

tion of functions such as  $\sum_{i=1}^n (Y_i - X_i \Theta)^2$ , where  $Y_i$  and  $X_i$  are the dependent and independent observations and  $\Theta$  is the parameter in question. The distance measure associated with this approach does not meet the triangle inequality and consequently the geometry of the analysis space is nonmetric (Mielke 1986). The data space of essentially all ecological studies (which may involve transformations) is usually perceived as an Euclidean space of measurements (e.g., biomass, height, cover and weight), the most common metric space. This difference between the geometry of the analysis space and the data space imply that the congruence principle (Mielke 1985, 1986) is not satisfied. This violation of the congruence principle is the root of commonly encountered contradictions between one's intuition (from observation) and inferential results based on well known statistical methods (such as ANOVA, MANOVA, etc.).

As was mentioned before, all of the traditional statistical methods are mean based methods. In many instances pertaining to ecology, the mean value may be less important than other parameters. In Fig. 2, for instance, both communities may have similar mean values for biomass, but this could be ecologically less

important than how this biomass is distributed over the landscape. As a consequence, a statistical procedure that tests for biomass concentration rather than averages would be more meaningful. In a normal distribution the mean and the median are equal. If the distribution of the data is both non-normal and non-symmetric, however, the median value in some circumstances could be ecologically more important than the mean value.

The above discussion clearly indicates that ecological and vegetation research is in need of a strong statistical inference method that (1) is free from assumptions about a normal (or multivariate normal) distribution, equality of variances (or variance-covariance matrices) and linear relationships; (2) satisfies the congruence principle (i.e., the analysis space is a Euclidean space); and (3) is flexible enough to allow for the testing of a variety of structural differences (means, medians, data concentration or dispersion, etc.) between the populations in question. In Section 2 we describe statistical methods, *multiresponse permutation procedures* (MRPP) and its *randomized block design* analog (MRBP), that meet the criteria proposed above. In this section we also discuss the interconnection of MRPP and MRBP with ANOVA and MANOVA tests when nonmetric analysis spaces are involved. Section 3 gives an outline of algorithms and computer programs developed to implement MRPP and MRBP. Several examples with real and artificial data are presented in Section 4 to highlight major characteristics of MRPP and MRBP and how they compare with results derived from traditional statistical methods. A summary and comments about the advantages of MRPP and MPBP for the advances of ecological research is presented in Section 5. For those not interested in the mathematical foundations of MRPP and MRBP or the corresponding algorithms, we suggest that the reader proceed to Section 4.

## 2.0 The new statistical procedures

### 2.1 Methodological description (MRPP)

Let  $\Omega = [\omega, \dots, \omega_N]$  be a finite population of  $N$  objects, let  $X_I = (x_{I1}, \dots, x_{Ir})$  denote  $r$  commensurate response measurements (these might be functions of response measurements or residuals adjusted by predictors) for object  $\omega_I$  ( $I = 1, \dots, N$ ), and let  $S_1, \dots, S_{g+1}$  designate an exhaustive partitioning of the  $N$  objects comprising  $\Omega$  into  $g+1$  disjoint groups. The group  $s_{g+1}$  encompasses those objects that are not classified in the  $g$  groups, in other words, it represents an excess group. In most cases  $s_{g+1}$  will be empty but it can be very important in the testing of certain hypotheses as we will see in Section 4. Also let  $\Delta_{IJ}$  be a symmetric distance function value of the response measurements associated with the objects  $\omega_I$  and  $\omega_J$ . The statistic underlying MRPP is given by

$$\delta = \sum_{i=1}^g C_i \zeta_i \quad (2.1.1)$$

where

$$\zeta_i = \left( \frac{n}{2} \right)^{-1} \sum_{I < J} \Delta_{IJ} \Phi_i(\omega_I) \Phi_i(\omega_J) \quad (2.1.2)$$

is the average distance function value for all distinct pairs of objects in group  $S_i$  ( $i = 1, \dots, g$ ),  $n_i > 2$  is the number of *a priori* classified objects in group  $S_i$  ( $i = 1, \dots, g$ ),  $K = \sum_{i=1}^g n_i$ ,  $n_{g+1} = N - K$  is the number of remaining (unclassified) objects in the excess group  $S_{g+1}$ ,  $\sum_{I < J} \Delta_{IJ}$  is the sum over all  $I$  and  $J$  such that  $1 < I < J < N$ ,  $\Phi_i(\omega_I)$  is 1 if  $\omega_I$  belongs to  $S_i$  and 0 otherwise,  $C_i > 0$  ( $i = 1, \dots, g$ ), and  $\sum_{i=1}^g C_i = 1$ . The  $C_i$  value generally used is  $C_i = n_i/K$ . Discussed later is the impact of using alternative  $C_i$  values. The underlying permutation distribution of  $\delta$  (the null hypothesis) assigns equal probabilities to the

$$M = N! / \left( \prod_{i=1}^{g+1} n_i! \right)$$

possible allocations of the  $N$  objects to the  $g+1$  disjoint groups. The mean, variance and skewness of  $\delta$  under the null hypothesis are denoted respectively by  $\mu_\delta$ ,  $\sigma_\delta^2$ , and  $\gamma_\delta$  (Mielke, 1984). Under the null hypothesis, preliminary findings (Mielke et al. 1976; Mielke 1978, 1979) indicated some situations when the asymptotic distribution of  $N(\delta - \mu_\delta)$  is nondegenerate with the skewness of  $\delta$  being substantially negative. Based on results due to Sen (1970, 1972), O'Reilly and Mielke (1980) presented general theorems for the multivariate case of MRPP which characterize situations in which the distribution of  $N^{1/2}(\delta - \mu_\delta)$  is asymptotically normal under the null hypothesis. More recently, Brockwell et al. (1982) presented theorems for the univariate case of MRPP which delineate distributions for situations (probably the most important situations) in which the nondegenerate distribution of  $N(\delta - \mu_\delta)$  is not asymptotically normal under the null hypothesis. The multivariate generalizations of the results by Brockwell et al. (1982) and special situations analogous to those considered by Mielke and Sen (1981) for linear rank statistics are open questions which require further attention.

The symmetric distance function ( $\Delta_{IJ}$ ) is extremely important since it defines the structure of the underlying analysis space of MRPP. The form of the symmetric distance functions considered in this paper will be confined to

$$\Delta_{IJ} = \left( \sum_{k=1}^r |x_{kI} - x_{kJ}|^p \right)^{1/p} \quad (2.1.3)$$

where  $p > 1$  and  $v > 0$  ( $p$  is not relevant when  $r = 1$ ). In particular, the underlying analysis space of MRPP is nonmetric when  $v > 1$  (i.e., the triangle inequality property of a metric space fails) and is metric when  $v \leq 1$  (a distorted metric space when  $v < 1$ ). The analysis space of MRPP is a Euclidean space when  $p = 2$  and  $v = 1$ . While the validity of a permutation test is not affected by these geometric considerations, the rejection region of any test is highly dependent on the underlying geometry. Thus a geometry problem (i.e., either a nonmetric or a distorted metric space) will affect the power of a permutation test. The results of a permutation test will surely be misleading if the rejection region is incomprehensible. As a consequence, the choice of  $p = 2$  and  $v = 1$  is recommended for routine applications. This may be a controversial conclusion since, as subsequently demonstrated, the majority of permutation tests presently used in routine applications are based on  $v = 2$ .

Since the symmetric distance functions considered in this presentation are of the type described in the previous paragraph, small values of  $\delta$  imply a concentration of response measurements within the  $g$  groups. Thus the  $P$ -value for a realized value of  $\delta$  (say  $\delta_0$ ) is the probability statement given by  $P(\delta < \delta_0)$ .

As an example, let's look at a case where we have two groups of objects (A and B) from which two measured responses ( $x_1$  and  $x_2$ ) have been made (Fig. 3). The seven points of Figure 3 imply there are  $\binom{7}{2} = 21$  distinct pairs of points and consequently 21 distances must be computed. These 21 distances are listed in Table 1 and ordered from the lowest to highest value. The

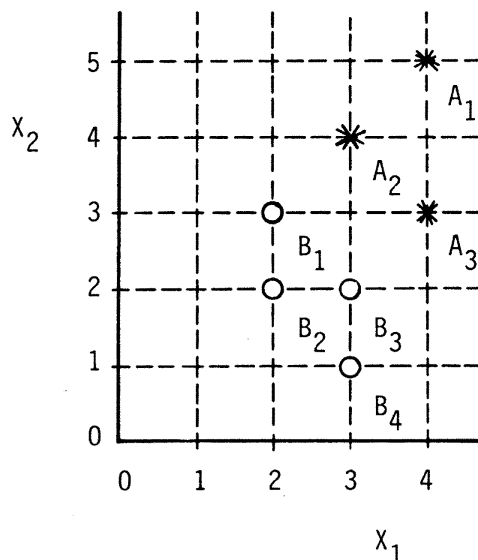


Fig. 3. Scatter diagram showing the points of the two subgroups (A and B) plotted as a function of the measured responses  $X_1$  and  $X_2$ .

Table 1. Ordered distances between all 21 pairs of the seven points shown in Figure 4 where distances between points in either subgroup A or subgroup B are indicated by crosses or circles, respectively.

Rank	Points	Distance
1	$B_1B_2$	1.000 (o)
2	$B_2B_3$	1.000 (o)
3	$B_3B_4$	1.000 (o)
4	$A_1A_2$	1.414 (x)
5	$A_2A_3$	1.414 (x)
6	$A_2B_1$	1.414
7	$A_3B_3$	1.414
8	$B_1B_3$	1.414 (o)
9	$B_2B_4$	1.414 (o)
10	$A_1A_3$	2.000 (x)
11	$A_2B_3$	2.000
12	$A_3B_1$	2.000
13	$A_2B_2$	2.236
14	$A_3B_2$	2.236
15	$A_3B_4$	2.236
16	$B_1B_4$	2.236 (o)
17	$A_1B_1$	2.828
18	$A_2B_4$	3.000
19	$A_1B_3$	3.162
20	$A_1B_2$	3.606
21	$A_1B_4$	4.123

distances  $\Delta_{I,J}$  are in this case calculated with the use of  $p = 2$  and  $v = 1$  in (2.1.3). Table 1 confirms the visual impression of clustering since the distance between points of a common group tend to be smaller than the distances between points of different groups. The values for  $\zeta_i$ ,  $\delta$  and  $M$  in this case are:

$$\zeta_A = (1/3) \sum_A \Delta_{I,J} = 1.6095$$

$$\zeta_B = (1/6) \sum_B \Delta_{I,J} = 1.3441$$

$$\delta = (3/7) \zeta_A + (4/7) \zeta_B = 1.4578$$

$$M = 7!/(3! 4!) = 35$$

Since  $M = 35$  for this example, 35 values of  $\delta$  can be obtained by enumerating all the possible 35 partitions. These 35 values of  $\delta$  are listed in Table 2 are ordered from the lowest to highest value. We see that the observed statistic ( $\delta_0 = 1.4578$ ) obtained for the realized partition (A and B) is indeed unusual since each of the remaining 34 values is greater than  $\delta_0$ . If all partitions could have occurred with equal chance (the null hypothesis), then the observed significance level or  $P$ -value is  $1/35 = 0.0286$ . Thus, we would accept the realized partition (A and B) as significant at the 2.8% level of significance.

Although an efficient algorithm for calculating the exact  $P$ -value for an observed value of  $\delta$  has been developed (Berry and Mielke 1984), this procedure is extremely expensive when  $M$  is large (e.g.,  $M > 10^6$ ).

**Table 2. Ordered values of  $\delta$  for all 35 partitions of the seven points shown in Figure 4 into two groups (A and B) having fixed sizes  $n_A = 3$  and  $n_B \leq 4$ .**

Rank	Value	Rank	Value
1	1.4578	19	2.1381
2	1.5421	20	2.1480
3	1.6939	21	2.1591
4	1.7505	22	2.1646
5	1.8389	23	2.1709
6	1.8547	24	2.1740
7	1.8935	25	2.1769
8	1.9898	26	2.1891
9	1.9915	27	2.1939
10	1.9988	28	2.2025
11	2.0060	29	2.2169
12	2.0157	30	2.2258
13	2.0176	31	2.2280
14	2.0522	32	2.2470
15	2.0575	33	2.2518
16	2.0829	34	2.2812
17	2.0944	35	2.2935
18	2.1158		

Since  $n_1 = n_2 = 12$  and  $N = 24$  yields  $M = 2.7 \times 10^6$  or  $n_1 = n_2 = n_3 = 6$  and  $N = 18$  yields  $M = 1.7 \times 10^7$ , the need for an approximation procedure becomes essential for cases involving even relatively small sample sizes. Because the distribution of  $\delta$  under the null hypothesis may differ substantially from a normal distribution for either small, moderate or extremely large sample sizes (Mielke 1978, 1979; O'Reilly and Mielke 1980; Brockwell et al. 1982), approximate P-values are based on the Pearson type III distribution which compensates for the fact that the underlying permutation distribution is often substantially skewed (Harter 1969; Mielke et al. 1981). In particular, the standardized test statistic given by

$$T = (\delta_0 - \mu_\delta) / \sigma_\delta$$

is presumed to follow the Pearson type III distribution with the density function given by

$$f(y) = \frac{(-2\gamma)^{4/\gamma^2}}{\Gamma(4/\gamma^2)} [-(2 + y\gamma) / \gamma]^{(4-\gamma^2)/\gamma^2} e^{-2(2+y\gamma)/\gamma^2}$$

where  $-\infty < y < -2/\gamma$ . If  $T_0 = (\delta_0 - \mu_\delta) / \sigma_\delta$  and  $\gamma = \gamma_\delta < -0.001$ , then

$$P(\delta < \delta_0) = \int_{-\infty}^{T_0} f(y) dy$$

is the approximate P-value (an approximate P-value based on the standard normal distribution is reported if  $\gamma_\delta > -0.001$ ). The approximate P-value is evaluated with Simpson's rule over the interval  $(T_0 - 9, T_0)$ .

Incidentally, the expressions given by  $\sum_{i < j} \Delta_{i,j} \Phi_i(\omega_i) \Phi_j(\omega_j)$  are the primary group random variables of tests based on statistic  $\delta$ . Let the symmetric dis-

ce functions be restricted to

$$\Delta_{i,j} = |x_i - x_j|^v \quad (2.1.4)$$

where  $v > 0$ . Also suppose  $x_{1,n} \leq x_{2,n} \leq \dots \leq x_{n,n}$  denote the  $n$  order statistics associated with the response measurements for a specific group of size  $n$ . Then the group random variable in question here is given by  $\sum_{i < j} |x_{i,n} - x_{j,n}|^v$  where  $\sum_{i < j}$  is the sum over all  $i$  and  $j$  such that  $1 < i < j < n$ . If  $v = 1$ , then

$$\sum_{i=1}^n |n - 2i + 1| |x_{i,n} - \Phi| = \sum_{i < j} |x_{i,n} - x_{j,n}| \quad (2.1.5)$$

where both equation (2.1.5) is satisfied and the left-hand side of equation (2.1.5) is minimized when  $\Phi$  is the median of  $x_{1,n}, \dots, x_{n,n}$ . Therefore tests based on  $v = 1$  are termed median based permutation techniques. Similarly, if  $v = 2$ ,

$$\sum_{i=1}^n (x_{i,n} - \Phi)^2 = \sum_{i < j} (x_{i,n} - x_{j,n})^2 \quad (2.1.6)$$

where both equation (2.1.6) is satisfied and the left-hand side of equation (2.1.6) is minimized when  $\Phi$  is the mean of  $x_{1,n}, \dots, x_{n,n}$ . As a consequence, tests based on  $v = 2$  are termed mean based permutation techniques.

## 2.2 Relationship between MRPP and other well known methods

We first describe the relationship between the permutation version of one-way analysis of variance (two-sample  $t$  test when  $g = 2$ ) and MRPP. Let  $F = MS_A / MS_W$  (mean square among groups/mean square within groups) be the ordinary one-way analysis of variance statistic. If  $g \geq 2$ ,  $v = 2$ ,  $r = 1$ ,  $A = \sum_{i=1}^N x_{iI}$ ,  $B = \sum_{i=1}^N x_{iII}^2$ ,  $N = K$  and  $C_i = (n_i - 1)/(N - g)$  for  $i = 1, \dots, g$ , then the identity relating  $F$  and  $\delta$  is given by

$$N\delta = 2(NB - A^2) / [N - g + (g - 1)F].$$

Because  $F$  is based on  $v = 2$ , the previously mentioned geometry problem of the underlying analysis space is a relevant concern for the permutation version of one-way analysis of variance. In Section 4 we will see by a series of examples how this geometry problem is crucial in determining P-values under certain circumstances.

The relation between simple two-sample linear rank tests and MRPP is now described. Let  $H = \sum_{i=1}^N f_I U_i$  be a two-sample linear rank test statistic where  $U_i = 1$  if  $\omega_i$  belongs to  $S_1$  and 0 otherwise,  $n_1 = n$ ,  $n_2 = N - n$ ,  $g = 2$ ,  $r = 1$ ,  $f_I$  is a score function of the rank order value of  $x_{iI}$  from below relative to the finite population of  $N$  response measurements,  $A = \sum_{i=1}^N f_I$ ,  $B$

$= \sum_{i=1}^N f_i^2$  and  $C_i = (n_i - 1)/(N - 2)$  for  $i = 1$  and  $2$ . Here the identity relating  $H$  and  $\delta$  is given by

$$N(N - 2)\delta = 2 \{NB - A^2 - (NH - nA)^2/[n(N - n)]\}.$$

Since  $H$  is also based on  $v = 2$ , the geometry problem of underlying analysis space is again a relevant concern.

Whaley (1983) demonstrates the equivalence of (a) the multidimensional runs statistic developed by Friedman and Rafsky (1979), (b) the spatial autocorrelation statistic introduced by Cliff and Ord (1973), and (c) the special case of the MRPP statistic when  $n_1 = n$ ,  $n_2 = N - n$ ,  $g = 2$ ,  $C_i = n_i(n_i - 1)/[n_1(n_1 - 1) + n_2(n_2 - 1)]$  for  $i = 1$  and  $2$ , and  $\Delta_{i,j}$  is  $1$  if  $\omega_i$  and  $\omega_j$  are linked and  $0$  otherwise. When  $n_1 = n_2$ , this interesting observation by Whaley (1983) suggests that the performance of this statistic might be improved if the present choice of  $C_i$  is replaced with  $C_i = n_i/N$  for  $i = 1$  and  $2$  (the simple structure of  $\Delta_{i,j}$  eliminates the geometry problem). As indicated by Mielke (1986), these tests usually involve a nonmetric analysis space.

### 2.3 Methodological description of MRBP

The techniques of this section involve recently developed analogs of MRPP which utilize blocking (Mielke and Berry 1982; Mielke and Iyer 1982). This section is followed by a description of the relationship between MRBP and well known statistical techniques that involve randomized block designs.

Let  $b$  blocks and  $g$  treatments be associated with a randomized block design. Let  $(x_{1ij}, \dots, x_{rij})$  denote  $r$  commensurate response measurements corresponding to treatment  $i$  and block  $j$  (the response measurements might again be functions of response measurements or residuals adjusted by predictors). The modified MRPP statistic for this situation is given by

$$\delta = [g \binom{b}{2}]^{-1} \sum_{i=1}^g \sum_{j < k} \Delta(x_{ij}, x_{ik})$$

where  $\Delta(x, y)$  is a symmetric distance function value of the points  $x' = (x_1, \dots, x_r)$  and  $y' = (y_1, \dots, y_r)$  in the  $r$ -dimensional Euclidean space. The underlying permutation distribution of  $\delta$  (the null hypothesis) assigns equal probabilities to be

$$M = (g!)^b$$

possible allocations of the  $g$   $r$ -dimensional response measurements to the  $g$  treatment positions within each of the  $b$  blocks. The mean, variance and skewness of  $\delta$  under the null hypothesis are again denoted by  $\mu_\delta$ ,  $\sigma_\delta^2$  and  $\gamma_\delta$ , respectively (Mielke, 1984). Except for subsequently described special cases when  $\delta$  is equivalent to well known statistics, little is presently known about

the asymptotic distribution of  $\delta$ . However, under the null hypothesis and fairly reasonable conditions, it is conjectured that (1)  $g^{1/2}(\delta - \mu_\delta)$  is asymptotically a normal random variable when  $b \geq 2$  is fixed and  $g \rightarrow \infty$ , and (2)  $b(\delta - \mu_\delta)$  is asymptotically a nondegenerate and non-normal random variable with  $\gamma_\delta < 0$  when  $g \geq 2$  is fixed and  $b \rightarrow \infty$ .

The symmetric distance function is again confined to

$$\Delta(x, y) = \left( \sum_{h=1}^r |x_h - y_h|^p \right)^{v/p} \quad (2.3.1)$$

where  $p > 1$  and  $v > 0$ . Since the choice of the symmetric distance function defines the structure of the underlying analysis space of these procedures, the discussion in Section 2.1 concerning this choice is equally pertinent here.

In a manner analogous to MRPP, small values of  $\delta$  imply a concentration of the response measurements associated with each of the  $g$  treatments (i.e., over blocks). Therefore  $P(\delta < \delta_0)$  is again the  $P$ -value associated with  $\delta_0$  (the realized value of  $\delta$ ). Though an efficient algorithm for calculating the exact  $P$ -value for an observed value of  $\delta$  exists, this approach becomes prohibitively expensive when  $M$  is large (e.g. greater than  $10^6$ ). Noting that  $b = 6$  and  $g = 4$  yields  $M = 1.9 \times 10^8$  or that  $b = 4$  and  $g = 6$  yields  $M = 2.7 \times 10^{11}$ , the necessity for an approximation technique is obvious for even relatively small randomized block configurations.

As in Section 2.1, approximate  $P$ -values are again based on the Pearson type III distribution to compensate for the commonly encountered substantial skewness of the underlying permutation distribution of  $\delta$ . Thus the standardized test statistic given by

$$T = (\delta - \mu_\delta) / \sigma_\delta$$

is again presumed to follow the Pearson type III distribution and the approximate  $P$ -value is obtained by the previously described approach in Section 2.1.

For certain response patterns involving randomized block designs, the statistic  $\delta$  mentioned above is not able to detect treatment differences. Such situations can occur when the block differences exceed the magnitude of the treatment differences. As a simple example, consider the following univariate data set ( $c = 1$ ) with  $b = 2$  and  $g = 3$ .

	Treatment		
	1	2	3
Block 1	1	2	3
Block 2	5	6	7

If  $v = 1$ , then  $\delta = 4$  for all  $M = 36$  possible allocations.

It is therefore impossible to detect treatment differences. This disturbing behavior of  $\delta$  can be rectified by aligning the responses within each block. The importance of alignment was initially pointed out by Hodges and Lehmann (1962). Alignment is accomplished for the present example by replacing  $x_{ij}$  by  $x_{ij} - x_j^*$  where  $x_j^*$  is the median of  $x_{1j}, \dots, x_{gj}$  for  $j = 1, \dots, b$ . Similarly  $x_j^*$  could just as well have been the mean of the responses in the  $j$ th block. The statistic  $\delta$  is then computed after the alignment. After alignment with the present example, the following data set is obtained.

	Treatment		
	1	2	3
Block 1	-1	0	1
Block 2	-1	0	1

Now the value of  $\delta$  is 0 while the random variable  $\delta$  assumes the values 0, 2/3 and 4/3 with the respective probabilities 1/6, 1/3 and 1/2 under the null hypothesis.

A general recommendation is that a test associated with  $\delta$  should be accomplished after alignment. Also note that if  $v = 2$  and  $r = 1$ , then the inferential results based on the random variable  $\delta$  remain unaffected by the alignment. This latter property holds because the  $F$  statistic does not depend on the values of  $x_{1j}^*, \dots, x_{bj}^*$ .

#### 2.4. Relationship between MRBP and other randomized block statistical methods.

The relation between  $\delta$  and the classical  $F$  statistic for testing the null hypothesis of a randomized block design is initially described. If  $v = 2$  and  $r = 1$ , then the functional relation between  $F$  and  $\delta$  is given by

$$F = (b-1) [2SS_T - g(b-1)$$

$$\delta] / [g(b-1)$$

$$\delta - 2SS_B]$$

where the corrected total sum of squares is given by  $SS_T = (\sum_{i=1}^g \sum_{j=1}^b x_{ij}^2) - SS_M$ , the block sum of squares is given by  $SS_B = \{ \sum_{j=1}^b [(\sum_{i=1}^g x_{ij})^2 / g] \} - SS_M$ , and  $SS_M = (\sum_{i=1}^g \sum_{j=1}^b x_{ij})^2 / bg$ . Thus  $F$  and  $\delta$  are equivalent under the null hypothesis since  $SS_T$  and  $SS_B$  are invariant relative to the  $(g!)^b$  permutations of the response measurements. (For this and other cases involving univariate responses, the response measurement subscript is omitted, i.e.,  $x_{ij} = x_{ij}$ .) Incidentally,  $\delta$  is equivalent to Cochran's  $Q$  test statistic if  $r = 1$  and each  $x_{ij}$  is either 0 or 1.

Let  $R$  denote the ordinary Pearson correlation coefficient. If  $v = 2$ ,  $b = 2$  and  $r = 1$ , then the functional relation between  $R$  and  $\delta$  is given by

$$R = (\mu_\delta - \delta) / (2S_1 S_2)$$

where  $R = [\sum_{i=1}^g (x_{i1} - \bar{x}_1)(x_{i2} - \bar{x}_2)] / (gS_1 S_2)$ ,  $\mu_\delta = S_1^2 + S_2^2 + (\bar{x}_1 - \bar{x}_2)^2$ ,  $\bar{x}_j = (\sum_{i=1}^g x_{ij})/g$  and  $S_j^2 = [\sum_{i=1}^g (x_{ij} - \bar{x}_j)^2]/g$  for  $j = 1$  and  $2$ . Then  $R$  and  $\delta$  are equivalent under the null hypothesis because  $\bar{x}_1$ ,  $\bar{x}_2$ ,  $S_1$  and  $S_2$  are invariant relative to the  $(g!)^2$  response measurement permutations.

If  $g = 2$ ,  $r = 1$ ,  $x_{1j} = -x_{2j} = x_j$  and  $|x_j| > 0$  for  $j = 1, \dots, b$ , then the test based on  $\delta$  is equivalent to an extended class of permutation techniques for matched pairs (Mielke and Berry 1982).

If  $v = 2$ ,  $r = 1$  and the response measurements for each block are replaced by their corresponding ranks, then the test based on  $\delta$  is equivalent to the Friedman two-way analysis of variance (Kendall coefficient of concordance). The values of  $\mu_\delta$ ,  $\sigma_\delta^2$  and  $\gamma_\delta$  for this case are

$$\mu_\delta = (g^2 - 1)/6, \quad \sigma_\delta^2 = [(g+1)(g^2 - 1)]/[18b(b-1)],$$

and

$$\gamma_\delta = -\{8(b-2)^2/[(g-1)b(b-1)]\}^{1/2}.$$

Furthermore  $\rho = 1 - \delta/\mu_\delta$  is Spearman's rho (a measure of correlation) when  $b = 2$ . In the present context, the 'correlation' measure  $\rho = 1 - \delta/\mu_\delta$  can be interpreted in a much broader setting.

If  $v = r = 1$  and the response measurements for each block are again replaced by their corresponding ranks, then the test based on  $\delta$  is the Euclidean space analog of the Friedman two-way analysis of variance (Kendall coefficient of concordance). The values of  $\mu_\delta$ ,  $\sigma_\delta^2$  and  $\gamma_\delta$  for this case are

$$\mu_\delta = (g^2 - 1) / (3g), \quad \sigma_\delta^2 = \frac{2(g+1)(2g^2 + 7)}{45g^2b(b-1)},$$

and

$$\gamma_\delta = \frac{(g+2)(2g^2 + 31)\Theta(g) + (8g^4 + 29g^2 + 71)}{[49(g+1)(2g^2 + 7)]^3}$$

$$\frac{(b-2)/(g-1)}{b(b-1)/40]^{1/2}}$$

where  $\Theta(g) = 0$  or  $1$  if  $g = 2$  or  $g \geq 3$ , respectively. In this case  $\delta$  is the Spearman footrule statistic when  $b = 2$  (Diaconis and Graham 1977).

### 3.0 Implementation of the method

As mentioned in Section 2 for moderately large sample sizes ( $N > 24$ ) the Pearson type III distribution is used to estimate  $P$  values. For that purpose, it is necessary to compute values for  $\mu_\delta$ ,  $\sigma_\delta^2$  and  $\gamma_\delta$  (mean, variance

and skewness). In Sections 3.1 and 3.2 we outline efficient algorithms to estimate  $\mu_\delta$ ,  $\sigma_\delta^2$  and  $\gamma_\delta$  for both MRPP and MRBP. We also give a very brief description of the computer program designed to carry out these computations. The FORTRAN coding for the computer programs is given in Appendix I and II.

### 3.1 Algorithms for the MRPP statistical method

Efficient computational expressions for  $\mu_\delta$ ,  $\sigma_\delta^2$  and  $\gamma_\delta$  are given by:

$$\mu_\delta = D(1),$$

$$\sigma_\delta^2 = 2 \left\{ \sum_{i=1}^g C_i^2 [n_i^{(2)}]^{-1} - [N^{(2)}]^{-1} \right\} [D(2) - 2D(2') + D(2'')]$$

$$+ 4 \left[ \sum_{i=1}^g C_i^2 n_i^{-1} - N^{-1} \right] [D(2') - D(2'')],$$

$$\gamma_\delta = \{E[\delta^3] - 3\mu_\delta \sigma_\delta^2 - \mu_\delta^3\} / \sigma_\delta^3,$$

and

$$E[\delta^3] = 4 \sum_{i=1}^g C_i^3 [n_i^{(2)}]^{-2} D(3) + 8 \sum C_i^3 n_i^{(3)} [n_i^{(2)}]^{-3} [3D(3') + D(3^*)]$$

$$+ 8 \sum_{i=1}^g C_i^3 n_i^{(4)} [n_i^{(2)}]^{-3} [3D(3^{**}) + D(3^{***})]$$

$$+ 6 \sum_{i=1}^g C_i^2 \{1 - C_i + C_i n_i^{(4)} [n_i^{(2)}]^{-2}\} [n_i^{(2)}]^{-1} D(3''')$$

$$+ 12 \sum_{i=1}^g C_i^2 \{(1 - C_i) n_i^{(3)} + C_i n_i^{(5)} [n_i^{(2)}]^{-1}\} [n_i^{(2)}]^{-2} D(3''')$$

$$+ \sum_{i=1}^g C_i \{(1 - C_i)(1 - 2C_i) + 3C_i(1 - C_i) n_i^{(4)} [n_i^{(2)}]^{-2}\}$$

$$+ C_i^2 n_i^{(6)} [n_i^{(2)}]^{-3} \} D(3''')$$

where  $n^{(c)} = n!/(n - c)!$

The  $D(i)$  where  $i = 1, 2, 2', 2'', 3, 3', 3'', 3''', 3''', 3^*,$

$3^{**}$ , and  $3^{***}$  are calculated with the use of the following efficient algorithm: let

$$d_{kJ} = \sum_{j=1}^N \Delta_{J,j}^k \text{ and } d_k = \sum_{j=1}^N d_{kJ}$$

for  $k = 1, 2$  and  $3$ , where  $\Delta_{J,j}$  is calculated as in 2.1.3. Then

$$D(1) = \frac{1}{N^{(2)}} d_1, D(2) = \frac{1}{N^{(2)}} d_2, D(2') = \frac{1}{N^{(3)}}$$

$$\left[ \sum_{j=1}^N d_{1j}^2 - d_2 \right],$$

$$D(2'') = \frac{1}{N^{(4)}} [d_1^2 - 4N^{(3)} D(2') - 2d_2], D(3) =$$

$$\frac{1}{N^{(2)}} d_3,$$

$$D(3') = \frac{1}{N^{(3)}} \left[ \sum_{j=1}^N d_{1j} d_{2j} - d_3 \right], D(3'') = \frac{1}{N^{(4)}}$$

$$[d_1 d_2 - 4N^{(3)} D(3') - 2d_3],$$

$$D(3^*) = \frac{6}{N^{(3)}} \sum_{j_1 < j_2 < j_3} \Delta_{j_1, j_2} \Delta_{j_1, j_3} \Delta_{j_2, j_3},$$

$$D(3^{**}) = \frac{1}{N^{(4)}} [2 \sum_{j_1 < j_2} \Delta_{j_1, j_2} d_{1j_1} d_{1j_2} - 2N^{(3)}$$

$$D(3') - N^{(3)} D(3^*) - d_3],$$

$$D(3^{***}) = \frac{1}{N^{(4)}} \left[ \sum_{j=1}^N d_{1j}^3 - 3N^{(3)} D(3') - d_3 \right],$$

$$D(3''') = \frac{1}{N^{(5)}} [N^{(3)} d_1 D(2') - 4N^{(4)} D(3^{**}) -$$

$$2N^{(4)} D(3^{***}) - 4N^{(3)} D(3') - 2N^{(3)} D(3^*)],$$

and

$$D(3''''') = \frac{1}{N^{(6)}} [N^{(4)} d_1 D(2'') - 8N^{(5)} D(3''') -$$

$$4N^{(4)} D(3'') - 8N^{(4)} D(3^{**})]$$

where  $\sum_{j_1 < j_2} ( )$  denotes the summation over distinct integral values of  $j_1$  and  $j_2$ , ranging from 1 through  $N$  with  $j_1$  less than  $j_2$ , etc.

The computer program (see Appendix I for the de-



tailed FORTRAN coding) is designed to implement the above algorithms. Program DRIVER reads the required parameters and the variable format, calls subroutine MRPP, and prints the results of the MRPP analysis. Subroutine MRPP reads in the raw data, labels each grouping with a group number, calculates the test statistic (DELTA), the expected value of DELTA (MEAN), the variance of DELTA (SIGMA2), the skewness of DELTA (GAMMA 1), and then calls subroutine TYPE3 to obtain the P-value based on the Pearson type III distribution. Subroutine TYPE3 computes and returns the P-value based on the Pearson type III distribution using numerical integration involving the Newton-Cotes formula for a tenth order polynomial.

The program for MRPP in Appendix I has the following restrictions and constants.

LIMIT = the smallest power (−675.00) without an underflow condition occurring in function FOX3.

N = the number of partitions (500) over which the integration takes place in subroutine TYPE3. This value may be modified for increased/decreased accuracy/efficiency.

NINE = the number of standard deviations (9) over which the integration takes place in subroutine TYPE3.

TEST = selects the normal distribution if the absolute value of GAMMA1 is less than or equal to 0.001, otherwise, the Pearson Type III distribution is selected.

### 3.2 Algorithm for the MRBP statistical method

The computational expressions for  $\mu_\delta$ ,  $\sigma_\delta^2$  and  $\gamma_\delta$  in MRBP are given by

$$\mu_\delta = [g^2 \binom{b}{2}]^{-1} \sum_{r \in s} \text{SIJ}(r, s),$$

$$\sigma_\delta^2 = [g^2 \binom{b}{2}]^{-2} \frac{1}{g-1} \sum_{r \in s} \{ [\text{SIJ}(r, s)]^2 - g \text{TJ2}(r, s) + g^2 \text{SIJ2}(r, s) \},$$

$$\begin{aligned} K_3(\delta) = & [g^2 \binom{b}{2}]^{-3} \frac{1}{g-1} \left( \frac{1}{g-2} \sum_{r \in s} \{ 4[\text{SIJ}(r, s)]^3 \right. \\ & - 6g \text{SIJ}(r, s) \text{TJ2}(r, s) + 6g^2 \text{UIJ}(r, s) + 2g^2 \text{TIJ3}(r, s) \\ & + 3g^2 \text{SIJ}(r, s) \text{SIJ2}(r, s) - 3g^2 \text{VI}(r, s) + g^4 \text{SIJ3}(r, s) \} \\ & - \frac{6}{g-1} \sum_{r \in s \in t} \{ \text{SIJ}(r, s) \text{SIJ}(r, t) \text{SIJ}(s, t) \\ & - g [\text{SIJ}(s, t) \text{WI}(r, s, t) + \text{SIJ}(r, t) \text{WI}(s, r, t) \\ & + \text{SIJ}(r, s) \text{WI}(t, r, s)] + g^2 \text{YIJ}(r, s, t) \\ & \left. - g^3 \text{ZIJK}(r, s, t) \} \right), \end{aligned}$$

and

$$\gamma_\delta = K_3(\delta) / \sigma_\delta^3. \quad (2.5)$$

Let's define  $d_{ij}$  as a multivariate observation in treatment  $i$  block  $j$  and  $\Delta(i, r; j, s) = \Delta(d_{ir}, d_{js})$  where  $\Delta(x, y)$  is calculated as in 2.3.1. The quantities involved in the estimation of  $\mu_\delta$ ,  $\sigma_\delta^2$ , and  $\gamma_\delta$  are calculated in the following way:

$$\text{SJ}(i, r, s) = \sum_{j=1}^g \Delta(i, r; j, s),$$

$$\text{SJ2}(i, r, s) = \sum_{j=1}^g [\Delta(i, r; j, s)]^2,$$

$$\text{SJ3}(i, r, s) = \sum_{j=1}^g [\Delta(i, r; j, s)]^3,$$

$$\text{SIJ}(r, s) = \sum_{i=1}^g \text{SJ}(i, r, s),$$

$$\text{SIJ2}(r, s) = \sum_{i=1}^g \text{SJ2}(i, r, s),$$

$$\text{SIJ3}(r, s) = \sum_{i=1}^g \text{SJ3}(i, r, s),$$

$$\text{TJ2}(i, r, s) = [\text{SJ}(i, r, s)]^2 + [\text{SJ}(i, s, r)]^2,$$

$$\text{TJ3}(i, r, s) = [\text{SJ}(i, r, s)]^3 + [\text{SJ}(i, s, r)]^3,$$

$$\text{TIJ2}(r, s) = \sum_{i=1}^g \text{TJ2}(i, r, s),$$

$$\text{TIJ3}(r, s) = \sum_{i=1}^g \text{TJ3}(i, r, s),$$

$$\text{U}(i, r; j, s) = \Delta(i, r; j, s) \text{SJ}(i, r, s) \text{SJ}(j, s, r),$$

$$\text{UIJ}(i, r, s) = \sum_{j=1}^g \text{U}(i, r; j, s),$$

$$\text{UIJ}(r, s) = \sum_{i=1}^g \text{UIJ}(i, r, s),$$

$$\text{V}(i, r, s) = \text{SJ}(i, r, s) \text{SJ2}(i, r, s) + \text{SJ}(i, s, r) \text{SJ2}(i, s, r),$$

$$\text{VI}(r, s) = \sum_{i=1}^g \text{V}(i, r, s),$$

$$\text{W}(i; r, s, t) = \text{SJ}(i, r, s) \text{SJ}(i, r, t),$$

$$\text{WI}(r, s, t) = \sum_{i=1}^g \text{W}(i; r, s, t),$$

$$\begin{aligned} \text{Y}(i, j; r, s, t) = & \Delta(i, r; j, s) \text{SJ}(i, r, t) \text{SJ}(j, s, t) \\ & + \Delta(i, r; j, t) \text{SJ}(i, r, s) \text{SJ}(j, t, s) \\ & + \Delta(i, s; j, t) \text{SJ}(i, s, r) \text{SJ}(j, t, r), \end{aligned}$$

$$\text{YJ}(i; r, s, t) = \sum_{j=1}^g \text{Y}(i, j; r, s, t),$$

$$\text{YIJ}(r, s, t) = \sum_{i=1}^g \text{YJ}(i; r, s, t),$$

$$\text{Z}(i, j, k; r, s, t) = \Delta(i, r; j, s) \Delta(i, r; k, t) \Delta(j, s; k, t),$$

$$\text{ZK}(i, j; r, s, t) = \sum_{k=1}^g \text{Z}(i, j, k; r, s, t),$$

$$ZJK(i; r, s, t) = \sum_{k=1}^g ZK(i, j; r, s, t),$$

and

$$ZLJK(r, s, t) = \sum_{i=1}^g ZJK(i; r, s, t).$$

The computer program designed to implement this algorithm and calculate P consist of 3 main subroutines (for detail FORTRAN coding of MRBP see Appendix II). Subroutine MRBP reads in the raw data and calculates the distance functions, the test statistic, the mean, and variance, and the skewness of the sampling distribution. Acting as a driver program, subroutine MRBP then calls subroutines MRPP and PVALUE to calculate the probability value. Subroutine MRBP sequentially reads in the  $x_{irk}$  raw scores ( $i = 1, \dots, g; r = 1, \dots, b; k = 1, \dots, c$ ), calculates the required distance functions, and writes the computed values onto a random-access file for later retrieval. This subroutine also calculates  $\delta_0$  (as in 2.3.1),  $\mu_\delta$ ,  $\sigma_\delta$ , and  $\gamma_\delta$  (as outlined above) and  $\varrho$  (here  $\varrho = 1 - \delta/\mu_\delta$  is a generalized measure of agreement). It then computes the standardized test statistics  $T_0 = (\delta_0 - \mu_\delta)/\sigma_\delta$ .

#### 4.0 Examples

This section is devoted to show properties of MRPP and MRBP on different types of data sets (both artificial and experimentally derived). Comparisons are made between MRPP or MRBP and standard statistical method. A special emphasis is given to the relationships among the data space geometry, analysis space geometry, and derived P-values.

##### 4.1 Example 1

The first example (Table 3) shows a univariate case of two groups ( $S_1$  and  $S_2$ ) with equal means but different point concentrations (similar to the case of Fig. 2). This example can not be treated with a standard parametric test (let's say a one- or two-sided t test) because the variances are not homogeneous and thus violates one of the basic assumptions. But, even if the variances were homogeneous, a parametric test would be worthless because it only deals with location shifts in the mean and in this case the means are equal. Furthermore, even if the means were different and the varian-

ces homogeneous, a mean based test would be of little value since the means of  $S_2$  (5.19 in this case) have very little meaning as an interpretation tool because the points are concentrated in the extremes (in a sense, the mean stands in an empty space with no real data point close to it). The groups, regardless of their equal means, still show a totally different data structure.  $S_1$  is very concentrated while  $S_2$  is dispersed. This type of data set is particularly common in a comparison between a seeded pasture and a native pasture. Both could have the same biomass but a different distribution of the biomass over the landscape. For instance, the seeded pasture is bound to be homogeneous while the native one, especially in semiarid regions, is more likely to have the vegetation concentrated on clumps with many empty spaces. The MRPP and MRBP allows us to test for that without any concern about the underlying nature of the density functions involved. As such they give us an important tool to perform vegetation analysis which are not possible with other methods adding reliability to our results because no caveats are needed about normal distributions, linearity, homogeneity of variances, etc.

There are two ways to treat the problem presented in Table 3. One way involves a standard two group test ( $g=2$ ), in which case we test for the probability of having one group with a concentrated data structure and another with a dispersed one. The other way is to perform a one sample test ( $g=1$ ) and use the other group as an excess group (the  $g+1$  groups explained in Section 2). In this case, we will look at the probability of observing a given data point concentration when a sample of  $n$  individuals ( $n=11$  in this case) is drawn from a population of  $N$  ( $N=22$  in this case). Table 4 shows the test results. It can be seen from the results that there is great disparity between the use of  $v=1$  (both data and analysis spaces are metric spaces) and  $v=2$  (analysis space is a nonmetric space). As explained in the introduction, this is due to the fact that congruence principle is violated with  $v=2$  since the data space is a metric space but the analysis space is a nonmetric space. Furthermore when values of Table 3 are plotted in Fig. 4 and therefore shown in a metric space, it becomes intuitively obvious that  $S_1$  is more concentrated than  $S_2$ . This feature is captured by the analysis based on  $v=1$  but completely missed by the analysis with  $v=2$ . We showed in Sections 2.2 and 2.4 that MRPP and

Table 3. Two groups  $S_1$  and  $S_2$  with equal mean but different degree of point concentration.

Group	Data Points											Mean
$S_1$	5	6	4	5.5	6.2	5	5.2	5.1	4.9	5	5.2	5.19
$S_2$	15	12	0.1	0.1	15	0.5	0.2	0.42	0.1	13.5	0.17	5.19

MRBP can be related to other standard statistical methods when  $v=2$  is used. As a result this type of distortion is bound to be found in all standard statistical tests. We will cover this geometry problem more in detail in Example 4.

**Table 4. Two-sided P-value comparisons between groups S1 and S2 from Table 3 using MRPP with  $v=1$  and  $v=2$ . We also show a one sample MRPP for S1 and S2 with  $v=1$ .**

Test	P
MRPP S1 vs. S2 ( $v=1, g=2$ )	$0.247 \times 10^{-2}$
MRPP S1 vs. S2 ( $v=2, g=2$ )	0.81
S1 one sample MRPP ( $v=1, g=1$ )	$0.905 \times 10^{-4}$
S2 one sample MRPP ( $v=1, g=1$ )	0.99

#### 4.2 Example 2

This example (Table 5) is similar to Example 1 (Table 3) except that it was constructed in such a way as to make the two groups share not only an equal mean but also an equal median as well. By so doing, this case is untreatable not only by standard parametric procedures but also by most nonparametric procedures that are based on medians. Like Example 1, it is visually obvious that the values in  $S_1$  are again far more concentrated than the values in  $S_2$ . This structural characteristic may have very important ecological con-

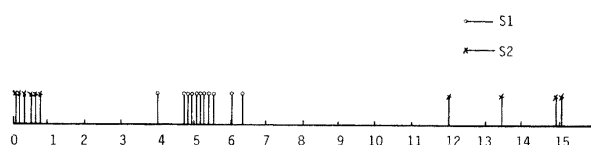
**Table 5. Two groups S1 and S2 with equal mean and median but different degrees of point concentration.**

Group	Data Points											Mean	Median
S1	4.1	4.1	4.2	4.5	4.6	4	3.9	3.98	3.99	3.89	3.92	4.10	4
S2	12	7	8	6	6	4	1	1	.01	.01	.16	4.10	4

acceptance may be intuitively wrong. The use of MRPP or MRBP with  $v=1$  gives an easy solution to this problem without having to either eliminate such values. Fig. 5 displays a scatter diagram of 20 bivariate observations partitioned into two groups,  $S_1$  and  $S_2$ , with one outlier with coordinates  $x_1=53$  and  $x_2=51$  belonging to  $S_1$ . By looking at Fig. 5, one would intuitively conclude that  $S_1$  and  $S_2$  are distinctively different. To test this hypothesis, we ran MRPP based on  $v=1$  and  $v=2$  and also four standard parametric procedures: Hotelling  $T^2$ , Pillai's criterion, Roy's maximum root and

**Table 6. Two-sided P-value comparison between groups S1 and S2 from Table 5 using MRPP with  $v=1$  and  $v=2$ .**

	MRPP $v=1$	MRPP $v=2$
S1 vs. S2	$0.27 \times 10^{-2}$	0.82



**Fig. 4. Plot for values on Table 3.**

sequences. MRPP allows us to investigate this type of data and the results are shown in Table 6. Again there is an enormous discrepancy between the p-values associated with  $v=1$  as compared to  $v=2$ .

#### 4.3 Example 3

One of the most common situations in vegetation research, or almost all biological research, is the presence of outliers (one or two points that seem not to agree with the general pattern of observed values). If these values are legitimate (in the sense that they are not errors in measurement), the investigator is confronted with the problem of how to treat the data. One way is to simply eliminate the data values in question from the analysis (which often goes unreported). The other way is simply to include the values in question (either as it is or with the use of some transformation) in the analysis. This inclusion (as we will see) may lead to the acceptance of the null hypothesis even though this

Wilk's lambda. Table 7 shows the results and once again there is an enormous difference between MRPP with  $v=1$  and the rest. Only MRPP based on  $v=1$  agrees with the intuitive observation. The reason is that, with the use of  $v=2$  in MRPP or the least squares procedure in the other four parametric tests, square distances are involved (that is our analysis space is a nonmetric space) and therefore the influence of outliers are made paramount. In the case of MRPP based on  $v=1$ , the congruence principle is upheld since the data space corresponds to the analysis space. The result is that one outlier does not overwhelm the remaining values and, as such, we have a better correspondence between our intuitive observations and the test results.

#### 4.4 Example 4

The intent of this example is to further consider the influence an individual outlier,  $w$ , when the remaining

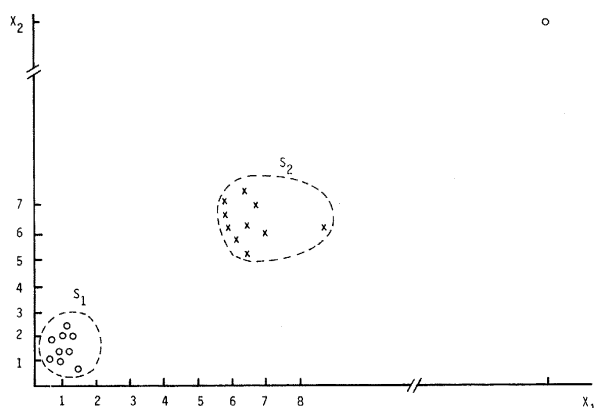


Fig. 5. Diagram showing points of two groups ( $S_1$  and  $S_2$ ) plotted as a function of 2 measured responses.  $o = S_1$ ,  $x = S_2$ .

influence an individual outlier,  $w$ , when the remaining values are specified. For this purpose, consider two groups  $S_1$  and  $S_2$  with  $n_1 = n_2 = n$ ,  $N = 2n$  and  $C_1 = C_2 = 1/2$ . Let  $n-1$  values associated with  $S_1$  be 0 and let the remaining value be  $w$ . Also let all  $n$  values associated with  $S_2$  be 1. For MRPP with  $v=1$  we have (using the algorithms of Section 3.1)

$$\delta = |\omega|/n,$$

$$\frac{n-2}{2n-1} + \frac{w}{n} \quad \text{if } w \geq 1,$$

$$\mu_\delta = \left(n - \frac{w}{n}\right) / (2n-1) \quad \text{if } 0 < w < 1,$$

$$\frac{n}{2n-1} - \frac{w}{n} \quad \text{if } w \leq 0,$$

and

$$\frac{(n+1)(n-2)}{(2n-3)} \quad \text{if } w \geq 1,$$

Table 7. P-values for MRPP with  $v=1$  and  $v=2$ , Hotelling  $T^2$ , Pillia's criterion, Roy's maximum root and Wilk's lambda on the data shown in Figure 5.

Test Procedure	P-value
MRPP, $v=1$	$0.65 \times 10^{-4}$
MRPP, $v=2$	0.99
Hotelling $T^2$	0.99
Pillia's criterion	0.99
Roy's maximum root	0.99
Wilk's lambda	0.99

$$\sigma_\delta^2 = \left(\frac{n-4w}{2n-3} + \frac{2w^2}{n-1}\right) / (2n-1)^2 \quad \text{if } 0 < w < 1,$$

$$n / [(2n-1)^2 (2n-3)] \quad \text{if } w \leq 0,$$

then statistic  $T = (\delta - \mu_\delta) / \sigma_\delta$  is

$$- [(n-1)(n-2)(2n-3) / (n+1)]^{1/2} \quad \text{if } w \geq 1,$$

$$T = - (n-2w) / \left(\frac{n-4w}{2n-3} + \frac{2w^2}{n-1}\right)^{1/2} \quad \text{if } 0 < w < 1, \quad (4.1.1)$$

$$- [n(2n-3)]^{1/2} \quad \text{if } w \leq 0.$$

Similarly for MRPP with  $v=2$  we have

$$\delta = w^2/n,$$

$$\mu_\delta = \frac{n-2w}{2n-1} + \frac{w^2}{n},$$

and

$$\sigma_\delta^2 = \left(\frac{n-4w}{2n-3} + \frac{2w^2}{n-1}\right) / (2n-1)^2,$$

then

$$T = - (n-2w) / \left(\frac{n-4w}{2n-3} + \frac{2w^2}{n-1}\right)^{1/2}. \quad (4.1.2)$$

If  $n$  is fixed, equation (4.1.1) shows that for the case of  $v=1$ ,  $T$  remains practically unchanged for all values of  $w$ . This indicates that the outlier will have a minimum influence in the statistical outcome when  $v=1$  since  $T$  is the basic statistics used to calculate the P-value (see Section 2.1). However, for the same case with  $v=2$ , equation (4.1.2) implies that

$$T \rightarrow \begin{aligned} &+ [2(n-1)]^{1/2} && \text{when } w \rightarrow +\infty \\ &- [2(n-1)]^{1/2} && \text{when } w \rightarrow -\infty \end{aligned}$$

Thus a statistical inference involving  $v=2$  may be either reversed by an individual value (outlier) which is inconsistent with the remaining data (when  $w$  is a large positive value in this example) or weakened by an individual value which is consistent with the remaining data (when  $w$  is a large negative value in this example). As empirical evidence to support these results, univariate and a multivariate examples follow.

The univariate data set in Table 8 involves two

groups,  $S_1$  with 14 specified values and one arbitrary value (denoted by  $x$ ) and  $S_2$  with 15 specified values. The 14 specified values of  $S_1$  were selected to be smaller than the values of  $S_2$  but allow for a slight overlap. The data point  $x$  in group  $S_1$  was allowed to take values from 18.4 to 98.4. For each value of  $x$ , the two groups were tested for statistical differences with a test which satisfies the congruence principle (MRPP with  $v=1$ ) and with two tests which do not satisfy the congruence principle (MRPP with  $v=2$  and the standard parametric two-sample  $t$  test). The P-values associated with each of the tests are shown in Table 9. as can be

**Table 8. Frequencies of  $S_1$  and  $S_2$  values.**

Value	$S_1$	$S_2$
58.2	1	0
58.3	4	0
58.4	5	0
58.5	3	1
58.6	1	3
58.7	0	4
58.8	0	5
58.9	0	2
$x$	1	0

readily observed, the P-values of MRPP with  $v=1$  are only slightly affected by the different values of  $x$  while the P-values of both MRPP with  $v=2$  and the two-sample  $t$  test are highly dominated by the different values of  $x$ .

For the multivariate example, consider the data presented in Figure 5. After altering the value of the outlier in a symmetrical way (i.e., replace  $x_1=53$  and  $x_2=51$  with  $x_1=-53$  and  $x_2=-51$ ), the same analysis shown in Table 7 is accomplished again. In this case, the P-values for Hotelling  $T^2$ , Pilla's criterion, Roy's maximum root and Wilk's lambda drop from 0.99 to 0.19, and the P-value for MRPP based on  $v=2$  drops from 0.99 to  $0.14 \times 10^{-4}$  whereas the P-value for MRPP based on  $v=1$  remains essentially unchanged, as anticipated by the previous results. These results are remarkable since they demonstrate that the alteration of only one data point in a symmetrical way can reverse conclusions of tests not satisfying the congruence principle (i.e., accepting to reject the null hypothesis even though the basic data structure has not been changed).

#### 4.5 Example 5

In this section we present the application of MRPP and MRBP to real case studies in vegetation research. The first case (Table 10) comes from a study on secondary succession (Biondini *et al.* 1985). The hypothesis to test was that increased levels of soil disturbance (from Treatment 1 that consisted of mechanical remo-

**Table 9. P-value comparisons of MRPP with  $v=1$ , MRPP with  $v=2$ , and the two-sample  $t$  test.**

$x$	MRPP		Two-sample $t$ test
	$v=1$	$v=2$	
18.4	$6.6 \times 10^{-7}$	$4.2 \times 10^{-6}$	0.27
48.4	$6.6 \times 10^{-7}$	$4.1 \times 10^{-6}$	0.15
53.4	$6.6 \times 10^{-7}$	$4.5 \times 10^{-6}$	$5.7 \times 10^{-2}$
56.4	$6.6 \times 10^{-7}$	$5.4 \times 10^{-6}$	$2.3 \times 10^{-3}$
57.4	$6.6 \times 10^{-7}$	$4.9 \times 10^{-6}$	$1.8 \times 10^{-5}$
58.4	$6.6 \times 10^{-7}$	$1.9 \times 10^{-6}$	$4.9 \times 10^{-9}$
59.4	$9.8 \times 10^{-6}$	$1.5 \times 10^{-3}$	$2.0 \times 10^{-3}$
60.4	$9.8 \times 10^{-6}$	0.14	0.16
63.4	$9.8 \times 10^{-6}$	0.99	1.00
68.4	$9.8 \times 10^{-6}$	0.99	0.62
98.4	$9.7 \times 10^{-6}$	0.99	0.39

val of vegetation with minimal disturbance to A and B soil horizon to Treatment 4 which consisted of physically inverting the soil to a depth of 2 m) significantly alter the direction of secondary succession. The main patterns of vegetation succession in this study were given by the following changes through time: (1) perennial grasses (the dominant species being: *Agropyron riparium*, *A. smithii*, *Koeleria cristata*, *Oryzopsis hymenoides*, and *Stipa comata*), (2) perennial forbs (the dominant species being: *Sphaeralcea coccinea*, *Erigeron engelmannii*, *Phlox longifolia*, *Senecio multilobatus*, and *Trifolium gymnocarpon*), (3) annual forbs (the dominant species being: *Salsola iberica*), and (4) shrubs (the dominant species being: *Artemisia tridentata*, *Chrysothamnus nauseosus*, *C. viscidiflorus*, and *Gutierrezia sarothrae*).

The hypothesis was formally tested with MRPP. The percent relative cover of perennial grasses, perennial forbs, annual forbs, and shrubs shown in Table 8 were used as the multivariate observation which characterized the species composition of each treatment. Treatments were analyzed at two points in time: (1) one year into succession, and (2) six years into succession. Treatments 2, 3, and 4 were shown to be different (P-value = 0.035) from Treatment 1 one year into succession. No significant difference (P-value = 0.11) was found among Treatments 2, 3, and 4. Six years later, Treatments 1 and 3 were marginally similar (P-value = 0.090). Treatment 2 was different from Treatments 1 and 3 (P-value = 0.042) and Treatment 4 was different from Treatment 2 (P-value = 0.03). The hypothesis that increased levels of soil disturbance can alter the pattern of secondary succession was accepted because of the following: (1) the species composition of Treatments 2, 3 and 4 was not statistically different in the early stages of succession while six years later the species composition of Treatments 2, 3 and 4 was statistically

**Table 10.** Percent relative cover of the dominant plant species groups of each treatment on years 1 and 6 of secondary succession.

Species Group	Treatment 1		Treatment 2		Treatment 3		Treatment 4	
	Yr 1	Yr 6	Yr 1	Yr 6	Yr 1	Yr 6	Yr 1	Yr 6
Grasses	49.04	62.15	14.27	38.07	1.27	43.80	0.04	5.44
Perennial Forbs	33.90	32.86	13.39	21.05	1.14	25.31	1.48	11.79
Annual Forbs	14.63	2.54	66.12	2.29	91.56	21.04	92.62	6.57
Shrubs	2.32	2.17	5.55	37.71	5.91	9.25	5.40	76.86

**NOTE:**

Treatment 1: the vegetation was mechanically removed with minimal disturbance to topsoil (A and B horizons).

Treatment 2: the vegetation was mechanically removed and the topsoil scarified to a depth of 30 cm.

Treatment 3: topsoil and subsoil (C horizon) were removed to a depth of 1 m. The material was mixed together and replaced.

Treatment 4: two layers of 1-m of soil were removed and replaced in a reverse order with the second layer placed on the surface.

different, and (2) Treatment 4 became a shrub-dominated community while Treatments 1 and 3 and to a lesser extent Treatment 2 became grass-forb dominated communities.

The second case comes from a study associated with mine reclamation research (Redente *et al.* 1982). An area that had been shallowly disturbed was seeded with a combination of grasses, forbs, and shrubs (for details

see Redente *et al.* 1982) and subjected to six treatments. The treatments were (a) no fertilizer, (b) low fertilizer, (c) high fertilizer, (d) mulch and fertilizer, (e) mulch and low fertilizer, and (f) mulch and high fertilizer (Table 9). The experiment was organized in a randomized block design with 3 blocks (Table 9). The biomasses of three shrubs (*Atriplex canescens*, *Ceratoides lanata*, and *Ephedra viridis*) were used as the multivariate obser-

**Table 11.** Biomass data on shrubs (g/m<sup>2</sup>) for a reclamation study on a shallowly disturbed site (Redente *et al.* 1982). The three shrubs measured on each treatment (listed from top to bottom) are: *Atriplex canescens*, *Ephedra viridis* and *Ceratoides lanata*. The treatments are (a) no fertilizer; b) low fertilizer (56 kg N/ha 28 kg P/ha); (c) high fertilizer (112 kg N/ha + 56 kg P/ha); (d) mulch (2.2 MT/ha of wood fiber hydromulch) and no fertilizer; (3) mulch and low fertilizer and (f) mulch and high fertilizer.

	Treatment					
	a	b	c	d	e	f
Block 1	0.33	6.67	6.33	3.83	9.67	14.50
	1.00	5.00	8.50	8.00	1.33	0.50
	2.17	2.00	2.17	1.33	3.67	2.17
Block 2	2.50	18.67	4.17	11.50	8.33	21.67
	0.83	0.17	1.67	2.50	1.50	0.83
	4.33	2.83	2.17	3.50	2.50	2.17
Block 3	1.00	8.67	5.17	0.67	23.67	7.33
	0.50	0.50	3.33	7.00	0.50	7.50
	2.33	2.00	2.00	2.50	3.33	4.50

vation that characterized each treatment (Table 9). The data was analyzed with the use of MRBP (based on  $v=1$ ) and, for comparison purposes with a series of standard multivariate parametric analyses (see Table 11). The null hypothesis (no difference between treatments) could be rejected with a P-value = 0.068 with MRBP based on  $v=1$  (which satisfies the congruence principle) but would have to be accepted if one were to use any of the other standard multivariate parametric methods (which do not satisfy the congruence principle).

### 5.0 Summary and conclusions

In the previous sections we explored the shortcomings inherent in most of the standard statistical parametric techniques when it comes to their application to ecological research in general and vegetation analysis in particular. With regard to the univariate (multivariate) parametric methods, a series of limitations had to do with the basic assumptions of normal (multivariate normal) distributions, homogeneity of variance (variance-covariance matrix) and linearity that have to

**Table 12. Results of MRBPP,  $v=1$  and other multivariate parametric tests run on the data shown on Table 10.**

Test Procedure	P-value
MRBPP based on $v=1$	0.068
Hotelling's criteria	0.21
Pillai's criterion	0.13
Roy's maximum root	0.20
Wilk's lambda	0.15

be made about the underlying structure of the population under study. Another potentially more serious limitation, which applies not only to parametric methods but also to most nonparametric methods, is the lack of congruence between the geometry of the data space and the geometry of the analysis space. While the data space on all of the vegetation and ecological studies is a well known metric space (a Euclidean space), the analysis space for most of the traditionally used statistical methods is a complex nonmetric space. As the examples that were presented have shown, this can lead to results that are both (1) inconsistent with intuitive observations and (2) unstable when only a single value is changed (see Example 4).

A new statistical method called multiresponse permutation procedures (MRPP) and its randomized block designed analog (MRBP) have been presented in detail in both their theoretical as well as practical foundations. Efficient algorithms as well as the associated computer programs needed to implement MRPP and MRBP were outlined in Sections 3.1 and 3.2. The method does not require any assumptions about the underlying distribution structure of the population under study and, more importantly, allows us to utilize a metric analysis

space. Furthermore, while the most commonly used parametric methods only test for shifts in the location of the means, MRPP and MRBP allow us to test for a variety of population structural differences (shifts in the location of the medians, changes in data concentration, etc.) which enlarges the scope of the hypotheses that can be accurately tested in vegetation analyses. We believe that MRPP and MRBP can be an important analytical tool that can (1) enlarge the scope of research studies in ecology and vegetation science and (2) be performed with a strict adherence to the hypothetical-deductive portion of the scientific method.

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[illegible]

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1      SIGMA2 = TWO * (S(1) - ONE / M(2)) * (D2 - TWO * D3 + D4) +
      FOUR * (S(2) - ONE / M(1)) * (D3 - D4)

      CALL TESTE THE SKEWNESS OF DELTA

DELT3 = FOUR * S(3) * D5 + EIGHT * S(4) * (THREE * D6 + D7) +
1      SIX * S(5) * D8 + EIGHT * S(6) * (THREE * D9 + D10) +
2      12.0 * S(7) * D11 + S(8) * D12
KAPPA3 = DELT3 - THREE * MEAN * SIGMA2 - MEAN ** 3
GAMMA1 = KAPPA3 / SIGMA2 ** 1.5

      CALL TYPE3 (DELTA, MEAN, SIGMA2, GAMMA1, PVALUE, IFAULT)

RETURN
END

SUBROUTINE TYPE3 (DELTA, MEAN, SIGMA2, GAMMA1, PVALUE, IFAULT)

DESCRIPTION OF VARIABLES IN CALL LIST:

DELTA - INPUT, OBSERVED (REALIZED) VALUE OF DELTA
MEAN - INPUT, EXPECTED (MEAN) VALUE OF DELTA
SIGMA2 - INPUT, VARIANCE OF THE DELTA DISTRIBUTION
GAMMA1 - INPUT, SKEWNESS OF THE DELTA DISTRIBUTION

PVALUE - OUTPUT, PROBABILITY OF A REALIZED DELTA VALUE THIS
EXTREME OR MORE EXTREME
IFAILT - OUTPUT, ZERO IF NO ERRORS

USAGE:

SUBROUTINE TYPE3 COMPUTES AND RETURNS A PROBABILITY VALUE
BASED ON THE PEARSON TYPE III PROBABILITY DISTRIBUTION.

REAL MEAN, NINE, LIMIT
PARAMETER (N = 500, LIMIT = 0.9999, TEST = 0.001)
DATA C1 / 16067.0/, C2 / 106300.0/, C3 / 48525.0/, C4 / 272400.0/,
1 C5 / 26050.0/, C6 / 427368.0/, C7 / 32194.0/, C8 / 299376.0/,
2 ZERO / 0.00/, ONE / 1.00/, TWO / 2.00/, THREE / 3.00/,
3 FOUR / 4.00/, FIVE / 5.00/, SEVEN / 7.00/, EIGHT / 8.00/,
4 NINE / 9.00/

TEST FOR THE NUMERICAL RANGE OF THE
GAMMA1 VALUE AND THE STANDARDIZED
TEST STATISTIC

T = (DELTA - MEAN) / SQRT (SIGMA2)
IF (ABS (GAMMA1) .GT. TEST) THEN
ALPHA = TWO / ABS (GAMMA1)
IF (GAMMA1 .LT. ZERO .AND. T .GT. LIMIT * ALPHA) THEN
PVALUE = ONE
ELSE IF (GAMMA1 .GT. ZERO .AND. T .LT. -LIMIT * ALPHA) THEN
PVALUE = ZERO
ELSE
GO TO 10
END IF
ELSE
CALCULATE A PROBABILITY VALUE BASED
ON THE NORMAL DISTRIBUTION

DATA B0 / 0.231641900/, B1 / 0.319381530/, B2 / -0.356563782/,
D = ONE / (ONE + B0 * ABS (T))
D = ONE / (ONE + B0 * ABS (T))
PI = FOUR * ATAN (ONE)
PVALUE = (EXP (-ONE * (T ** 2 / TWO))) / SQRT (TWO * PI)) *
1 D ** (B1 + D * (B2 - D * B4 + D * B5)))
END IF

CHECK ON VALUE OF IFAULT

IF (PVALUE .LT. ZERO) IFAULT = SEVEN
RETURN

CALCULATE THE VALUE OF THE NATURAL
LOG OF B

B = GHAT (ALPHA ** 2) + ALPHA ** 2 * (ONE - LOG (ALPHA))

BEGIN THE INTEGRATION ROUTINE

THETA = T
IF (GAMMA1 .LE. ZERO) THETA = THETA - NINE
C = (T - THETA) / FLOAT (N)
Y1 = C1 * FOX3 (ALPHA, GAMMA1, B, THETA)
Y2 = C1 * FOX3 (ALPHA, GAMMA1, B, THETA + NINE)
Y1 = ZERO
DO 20 I = 10, N, 10
Y2 = Y2 + C2 * (FOX3 (ALPHA, GAMMA1, B, THETA + (I - 9) * C)
1 + FOX3 (ALPHA, GAMMA1, B, THETA + (I - 1) * C)) - C3 *
2 (FOX3 (ALPHA, GAMMA1, B, THETA + (I - 8) * C) + FOX3 (
3 ALPHA, GAMMA1, B, THETA + (I - 2) * C)) + C4 * (FOX3 (
4 ALPHA, GAMMA1, B, THETA + (I - 7) * C) + FOX3 (ALPHA,
5 GAMMA1, B, THETA + (I - 3) * C)) - C5 * (FOX3 (ALPHA,
6 GAMMA1, B, THETA + (I - 6) * C) + FOX3 (ALPHA, GAMMA1,
7 B, THETA + (I - 4) * C)) + C6 * FOX3 (ALPHA, GAMMA1, B,
8 THETA + (I - 5) * C) + C7 * FOX3 (ALPHA, GAMMA1, B,
9 THETA + I * C)
20 CONTINUE

CALCULATE A PROBABILITY VALUE BASED
ON A PEARSON TYPE III DISTRIBUTION

PVALUE = FIVE * (T - THETA) * (Y0 - Y1 + Y2) / (C8 * N)

CHECK ON VALUE OF IFAULT

IF (PVALUE .LT. ZERO) IFAULT = EIGHT
RETURN
END

FUNCTION GHAT (Y)

DESCRIPTION OF VARIABLES IN CALL LIST:

Y - INPUT, VARIABLE PASSED FROM SUBROUTINE TYPE3

USAGE:

FUNCTION GHAT CALCULATES STIRLING'S APPROXIMATION FOR THE
NATURAL LOG OF (UPPER-CASE) GAMMA (Y).

DATA C1 / 12.0/, C2 / 360.0/, C3 / 1260.0/, C4 / 1680.0/,
1 C5 / 1188.0/, ZERO / 0.00/, ONE / 1.00/, TWO / 2.00/,
2 FOUR / 4.00/, TEN / 10.00/, HALF / 0.50/

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C      PI = FOUR * ATAN (ONE)
C      SUM = ZERO
C      X = Y + TEN
C      DO 10 I = 1, 10
C        SUM = SUM + LOG (X - I)
10 CONTINUE
C      GHAT = ((X - HALF) * LOG (X) - X + HALF * LOG (TWO * PI)) + ONE
C      1 / (C1 * X) - ONE / (C2 * X ** 3) + ONE / (C3 * X ** 5) -
C      2 ONE / (C4 * X ** 7) + ONE / (C5 * X ** 9)) - SUM
C      RETURN
C      END

C      FUNCTION FOX3 (ALPHA, GAMMA1, B, X)
C      DESCRIPTION OF VARIABLES IN CALL LIST:
C      ALPHA - INPUT, PARAMETER (ALPHA) PASSED FROM TYPE3
C      BETA - INPUT, PARAMETER (BETA) PASSED FROM TYPE3
C      GAMMA - INPUT, PARAMETER (GAMMA) PASSED FROM TYPE3

MRPP 496 C      B - INPUT, PARAMETER (B) PASSED FROM TYPE3
MRPP 497 C      X - INPUT, VARIABLE PASSED FROM SUBROUTINE TYPE3
MRPP 498 C      USAGE:
MRPP 499 C      FUNCTION FOX3 CALCULATES F(X) FOR THE PEARSON TYPE III
MRPP 500 C      PROBABILITY DISTRIBUTION.
MRPP 501 C
MRPP 502 C      REAL LIMIT
MRPP 503 C      PARAMETER (LIMIT = -673.0)
MRPP 504 C      DATA ZERO /0.00/, ONE /1.00/
MRPP 505 C
MRPP 506 C      FOX3 = ZERO
MRPP 507 C      IF (GAMMA1 .LT. ZERO) THEN
MRPP 508 C        POWER = (ALPHA ** 2 - ONE) * LOG (ALPHA - X) + ALPHA * X - B
MRPP 509 C      ELSE
MRPP 510 C        POWER = (ALPHA ** 2 - ONE) * LOG (ALPHA + X) - ALPHA * X - B
MRPP 511 C      END IF
MRPP 512 C      IF (POWER .GT. LIMIT) FOX3 = EXP (POWER)
MRPP 513 C      RETURN
MRPP 514 C      END
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## APPENDIX II

### FORTAN program for the MRBP statistical method

For details about the algorithms used see Section (3.2)

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PROGRAM MRBP
IMPLICIT REAL*8(A-H,O-Z)

C      THIS FORTAN PROGRAM COMPUTES THE TEST STATISTIC AND ASSOCIATED
C      P-VALUE FOR AN ANALYSIS OF A RANDOMIZED BLOCK EXPERIMENT (MRBP).
C      THE CORRESPONDENCE BETWEEN A CORRELATION ANALYSIS AND A
C      RANDOMIZED BLOCK EXPERIMENT CAN BE USED TO GET A CORRELATION
C      COEFFICIENT AS WELL. THE MAXIMUM VALUES OF G, B AND R CAN BE
C      CHANGED FOR ANY EXAMPLE. THE PRESENT MAXIMUM VALUES OF G, B
C      AND R IN THIS PROGRAM ARE RESPECTIVELY 10, 8 AND 5.

COMMON /BL1/D(80,80)
COMMON /BL2/SJ(10,8,8),SJ2(10,8,8),SJ3(10,8,8),UJ(10,8,8)
COMMON /BL3/V,G,B,R,DATA(10,8,5)
COMMON /BL4/WI(8,8,8),YIJ(8,8,8),ZJK(8,8,8)
COMMON /BL5/SIJ(8,8),SIJ2(8,8),SIJ3(8,8),UIJ(8,8),
1  TIJ2(8,8),TIJ3(8,8),VI(8,8)
COMMON /BL6/T,GAM,PROB
CHARACTER FMT*80,HEAD*80,FIN*64,FOUT*64
INTEGER G,B,R

C      THE DATA MATRIX MUST BE IN THE FOLLOWING SEQUENCE WITH EACH
C      OBJECT'S R RESPONSE VALUES ON A SEPARATE LINE AS FOLLOWS:
C      A(1,1,1),A(1,1,2),....A(1,1,R)
C      A(1,2,1),A(1,2,2),....A(1,2,R)
C      ....
C      A(1,B,1),A(1,B,2),....A(1,B,R)
C      A(2,1,1),A(2,1,2),....A(2,1,R)
C      ....
C      A(2,B,1),A(2,B,2),....A(2,B,R)
C      ....
C      A(G,1,1),A(G,1,2),....A(G,1,R)
C      ....
C      A(G,B,1),A(G,B,2),....A(G,B,R)

C      THE INPUT DATA ARE IN FREE FORMAT. SPECIFICALLY, V IS DISTANCE EXPONENT,
C      G IS # OF GROUPS, B IS # OF BLOCKS, AND R IS # OF RESPONSES.

WRITE(*,590)
590 FORMAT(' ',ANALYSIS OF RANDOMIZED BLOCK EXPERIMENT (MRBP)')
1  ' ' ENTER NAME OF INPUT DATA FILE: '\ '
READ(*,*)FIN
WRITE(*,591)
591 FORMAT(' ',ENTER NAME OF OUTPUT/DEVICE FILE: '\ ')
READ(*,*)FOUT
OPEN(5,FILE=FIN)
OPEN(6,FILE=FOUT,STATUS='NEW')
READ(5,*)HEAD
READ(5,*)V,G,B,R
WRITE(6,600)HEAD,V,R,G,B
600 FORMAT(' ',ANALYSIS OF RANDOMIZED BLOCK EXPERIMENT (MRBP): '
1  4X,A/4X,'DISTANCE EXPONENT: ',F4.1/4X,'WITH ',I5,
2  ' RESPONSES, ',I5,' GROUPS, ',I5,' BLOCKS.'
3  //1X,'** INPUT DATA: ')

DO 110 I=1,G
DO 107 J=1,B
READ(5,*,END=59)(DATA(I,J,K),K=1,R)
WRITE(6,58) I,J,(DATA(I,J,K),K=1,R)
107 CONTINUE
110 CONTINUE
58 FORMAT(2I5,8F10.2)
CLOSE(5)
CALL MRPP
GO TO 200
59 WRITE(6,60)
60 FORMAT(' ',ERROR--END OF FILE ENCOUNTERED')
WRITE(6,61) G,B,R
61 FORMAT(' ',CHECK THESE: G = ',I5,' B = ',I5,' R = ',I5)
200 STOP
END

SUBROUTINE MRPP
IMPLICIT REAL*8(A-H,O-Z)
COMMON /BL1/D(80,80)
COMMON /BL2/SJ(10,8,8),SJ2(10,8,8),SJ3(10,8,8),UJ(10,8,8)
COMMON /BL3/V,G,B,R,DATA(10,8,5)
COMMON /BL4/WI(8,8,8),YIJ(8,8,8),ZJK(8,8,8)
COMMON /BL5/SIJ(8,8),SIJ2(8,8),SIJ3(8,8),UIJ(8,8),
1  TIJ2(8,8),TIJ3(8,8),VI(8,8)
COMMON /BL6/T,GAM,PROB
INTEGER G,B,R

C      ONE=1.0D0
C      ZERO=0.0D0
C      X=ONE*V/2
C      BC2=ONE*B*(B-1)/2
C      DO 10 I=1,G
C        DO 9 J=1,B
C          DO 8 K=1,G
C            LO=1
C            IF (I.EQ.K) LO=J
C            DO 7 L=LO,B
C              IJ=B*(I-1)+J
C              KL=B*(K-1)+L
C              D(IJ,KL)=ZERO
C              DO 6 M=1,R
C                D(IJ,KL)=D(IJ,KL)+(DATA(I,J,M)-DATA(K,L,M))**2
C              CONTINUE
C              D(IJ,KL)=D(IJ,KL)**X
C              D(KL,IJ)=D(IJ,KL)
C            CONTINUE
C          CONTINUE
C        CONTINUE
C      DO 30 IS=1,R
C        DO 29 IR=1,B
C          IF (IR.EQ.IS) GO TO 28
C          SIJ(IR,IS)=ZERO
C          SIJ2(IR,IS)=ZERO
C          SIJ3(IR,IS)=ZERO
C          DO 27 I=1,G
C            SJ(I,IR,IS)=ZERO
C            SJ2(I,IR,IS)=ZERO
C            SJ3(I,IR,IS)=ZERO
C            DO 26 J=1,G
C              IRR=(I-1)*B+IR
C              JSS=(J-1)*B+IS
C              SJ(I,IR,IS)=SJ(I,IR,IS)+D(I,IR,JSS)
C              SJ2(I,IR,IS)=SJ2(I,IR,IS)+D(I,IR,JSS)**2
C              SJ3(I,IR,IS)=SJ3(I,IR,IS)+D(I,IR,JSS)**3
C            CONTINUE
C            SIJ(IR,IS)=SIJ(IR,IS)+SJ(I,IR,IS)
C            SIJ2(IR,IS)=SIJ2(IR,IS)+SJ2(I,IR,IS)
C            SIJ3(IR,IS)=SIJ3(IR,IS)+SJ3(I,IR,IS)
C          CONTINUE
C        CONTINUE
C      DO 30 IT=1,R
C        IF (B.LE.2) GO TO 51
C        DO 50 IT=3,B
C          IT1=IT-1
C          DO 49 IS=2,IT1
C            IS1=IS-1
C            DO 48 IR=1,IS1
C              WI(IR,IS,IT)=ZERO
C              WI(IS,IR,IT)=ZERO
C              WI(IT,IR,IS)=ZERO
C              YIJ(IR,IS,IT)=ZERO
C              ZJK(IR,IS,IT)=ZERO
C            DO 47 I=1,G
C              WI(IR,IS,IT)=WI(IR,IS,IT)+SJ(I,IR,IS)*SJ(I,IR,IT)
C              WI(IS,IR,IT)=WI(IS,IR,IT)+SJ(I,IS,IR)*SJ(I,IS,IT)
C              WI(IT,IR,IS)=WI(IT,IR,IS)+SJ(I,IT,IR)*SJ(I,IT,IS)
C            DO 46 J=1,G
C              IRR=B*(I-1)+IR
C              JSS=B*(J-1)+IS
C              JTT=B*(J-1)+IT
C              ISS=B*(I-1)+IS
C              YIJ(IR,IS,IT)=YIJ(IR,IS,IT)+
C                D(I,IR,JSS)*SJ(I,IR,IT)*SJ(J,IS,IT)+
C                D(I,IR,JTT)*SJ(I,IR,IS)*SJ(J,IT,IS)+
C                D(I,ISS,JTT)*SJ(I,IS,IR)*SJ(J,IT,IR)
C            DO 45 K=1,G
C              KTT=(K-1)*B+IT
C              ZJK(IR,IS,IT)=ZJK(IR,IS,IT)+
C                D(I,IR,JSS)*D(I,IR,KTT)*D(J,ISS,KTT)
C          CONTINUE
C        CONTINUE
C      T2=T2+SIJ(IR,IS)*SIJ(IR,IT)*SIJ(IS,IT)-
C        SIJ(IS,IT)*WI(IR,IS,IT)*G-
C        SIJ(IR,IT)*WI(IR,IS,IT)*G-

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```

3      SIJ(IR,IS)=MI(IT,IR,IS)*G+
4      YIJ(IR,IS,IT)*G*G-ZIJK(IR,IS,IT)*G*G*G
48     CONTINUE
49     CONTINUE
50     CONTINUE
51     T2=T2*6/(G-1)
52     CONTINUE
53     DO 100 IS=2,B
54       IS1=IS-1
55       DO 99 IR=1,IS1
56         TIJ2(IR,IS)=ZERO
57         TIJ3(IR,IS)=ZERO
58         VI(IR,IS)=ZERO
59         UIJ(IR,IS)=ZERO
60         DO 98 I=1,G
61           TIJ2(IR,IS)=TIJ2(IR,IS)+SJ(I,IR,IS)*2+
62             SJ(I,IS,IR)*2
63           TIJ3(IR,IS)=TIJ3(IR,IS)+SJ(I,IR,IS)*3+
64             SJ(I,IS,IR)*3
65           VI(IR,IS)=VI(IR,IS)+SJ(I,IR,IS)*SJ2(I,IR,IS)+
66             SJ(I,IS,IR)*SJ2(I,IS,IR)
67           UIJ(IR,IS)=ZERO
68           DO 97 J=1,G
69             IRR=B*(I-1)+IR
70             JSS=B*(J-1)+IS
71             UJ(I,IR,IS)=UJ(I,IR,IS)+
72               D(IRR,JSS)*SJ(I,IR,IS)*SJ(J,IS,IR)
73           CONTINUE
74           UIJ(IR,IS)=UIJ(IR,IS)+UJ(I,IR,IS)
75         CONTINUE
76       CONTINUE
77     CONTINUE
78     T1=ZERO
79     EDEL=ZERO
80     VAR=ZERO
81     DO 200 IS=2,B
82       IS1=IS-1
83       DO 199 IR=1,IS1
84         IF (C.LE.2) GO TO 196
85         T1=T1+(SIJ(IR,IS)*3)*4-SIJ(IR,IS)*TIJ2(IR,IS)*G*G+
86           UJ(IR,IS)*G*G*G+TIJ3(IR,IS)*G*G*G*2+
87           SIJ(IR,IS)*SIJ2(IR,IS)*G*G*G*3-
88           VI(IR,IS)*G*G*G*G*3+SIJ3(IR,IS)*G*G*G*G*G
89       EDEL=EDEL+SIJ(IR,IS)
90       VAR=VAR+SIJ(IR,IS)*SIJ(IR,IS)-TIJ2(IR,IS)*G+SIJ2(IR,IS)*G*G
91     CONTINUE
92     IF (C.LE.2) GO TO 206
93     T1=T1/(G-2)
94     C1=ONE/(BC2*G*G)
95     C2=C1*C1
96     C3=C2*C1
97     EDEL=C1*EDEL
98     VAR=VAR*C2/(G-1)
99     GAM=C3*(T1-T2)/(G-1)
100    GAM=GAM/DSQRT(VAR*3)
101    DELTA=ZERO
102    DO 300 IS=2,B
103      IS1=IS-1
104      DO 299 IR=1,IS1
105        DO 298 I=1,G
106          IRR=(I-1)*B+IR
107          ISS=(I-1)*B+IS
108          DELTA=DELTA+D(IRR,ISS)
109      CONTINUE
110    CONTINUE
111    CO=BC2*G
112    DELTA=DELTA/CO
113    T=(DELTA-EDEL)/DSQRT(VAR)
114    RHO=ONE-(DELTA/EDEL)
115    WRITE (6,590)
590   FORMAT (// ' ' RESULTS OF MRBP ANALYSIS: ' ')
116   WRITE (6,600) DELTA
600   FORMAT (11X, 'DELTA(D)', T29, ' ', G16.7)
117   WRITE (6,610) EDEL
610   FORMAT (11X, 'EXPECTED DELTA', T29, ' ', G16.7)
118   WRITE (6,620) VAR
620   FORMAT (11X, 'VARIANCE OF DELTA', T29, ' ', G16.7)
119   WRITE (6,630) GAM
630   FORMAT (11X, 'SKEWNESS OF DELTA', T29, ' ', G16.7)

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WRITE (6,640) RHO
640  FORMAT (11X, 'AGREEMENT MEASURE', T29, ' ', G16.7)
WRITE (6,650) T
650  FORMAT (11X, 'D-E[D]/STDEV(D)', T29, ' ', G16.7)
CALL PVALUE
WRITE (6,670) PROB
670  FORMAT (11X, 'P-VALUE', T29, ' ', G16.7)
RETURN
END

C
SUBROUTINE PVALUE
C
C   SUBROUTINE PVALUE CALCULATES THE PROBABILITY OF A
C   VALUE OF T BEING LESS THAN OR EQUAL TO THE OBSERVED
C   VALUE OF T . . . .
C
C   IMPLICIT REAL*8(A-H,O-Z)
C   COMMON /BL6/T,GAM,PROB
C
ONE = 1.0D0
ZERO = 0.0D0
PI = DATAN(ONE)*4
IF (DABS(GAM).LT.0.01D0) GO TO 50
R = ONE*2/DABS(GAM)
D = R*R
F = R*R+10
DO 10 I = 1,9
  D = D*(R*R+1)
10 CONTINUE
U = (F*2-1)*DLOG(F)/2-F*DLOG(PI*2)/2-DLOG(D)+ONE/(F*12)
& -ONE/(F*F*F*360)
A = R*R-1
B = R*R*(DLOG(R)-1)-U
G1 = 0.045D0
G2 = 0.09D0
G3 = 0.015D0
H1 = ZERO
H2 = ZERO
W = -1.99D0/GAM
IF (GAM.LT.ZERO) GO TO 30
IF (T.LT.W) THEN
  PROB = ZERO
  GO TO 60
END IF
X = T
Y = T+9
DO 20 I = 1,99
  H1 = H1+DEXP(A*DLOG(R+X+G1*(2*I-1))-R*(X+G1*(2*I-1))+B)
  H2 = H2+DEXP(A*DLOG(R+X+G2*I)-R*(X+G2*I)+B)
20 CONTINUE
H1N = DEXP(A*DLOG(R+X+G1*199))-R*(X+G1*199)+B
H0 = DEXP(A*DLOG(R+X)-R*X+B)
H3 = DEXP(A*DLOG(R+Y)-R*Y+B)
PROB = ONE-G3*(H0+(H1+H1N)*4+H2*2+H3)
GO TO 60
30 IF (T.GT.W) THEN
  PROB = ONE
  GO TO 60
END IF
X = T-9
Y = T
DO 40 I = 1,99
  H1 = H1+DEXP(A*DLOG(R-X-G1*(2*I-1))+R*(X+G1*(2*I-1))+B)
  H2 = H2+DEXP(A*DLOG(R-X-G2*I)+R*(X+G2*I)+B)
40 CONTINUE
H1N = DEXP(A*DLOG(R-X-G1*199))+R*(X+G1*199)+B
H0 = DEXP(A*DLOG(R-X)-R*X+B)
H3 = DEXP(A*DLOG(R-Y)+R*Y+B)
PROB = G3*(H0+(H1+H1N)*4+H2*2+H3)
GO TO 60
50 E1 = 0.31938153D0
E2 = -0.356563782D0
E3 = 1.781477937D0
E4 = -1.821255978D0
E5 = 1.330274429D0
H = 0.2316419D0
W = ONE/(H*DABS(T)+1)
PROB = (((E5*W+E4)*W+E3)*W+E2)*W+E1)*W*DEXP(-T*T/2)/DSQRT(PI*2)
IF (T.GT.ZERO) PROB = ONE-PROB
60 RETURN
END

```